Externalities in Networks

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Abstract

In network theory, externalities play a critical role in determining which networks are optimal. Adding links can create positive externalities, as they potentially make distant vertices closer. On the other hand, links can result in negative externalities if they increase congestion or add competition. This paper introduces two new equilibrium concepts and will completely characterize the set of optimal and equilibrium networks for a natural class of negative externalities models where an agent's payoff is a function of the degree of her neighbors. These results are in sharp contrast to the optimal and equilibrium networks for the standard class of positive externalities models where payoff is a function of the distance two agents are apart.

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1 Introduction

Networks have long been studied in sociology, computer science, physics and mathematics. However, economists have only recently begun to focus on networks. This is surprising as graphs are a natural model of many economic situations. Most people find a job through a network of friends and associates.[1] Similarly, a car manufacturer does not buy its brakes from a marketplace but rather has long term relationships with its suppliers.

Fortunately, networks have started to receive the attention they deserve.[2] Whether out of convenience or necessity, virtually all networking papers employ a reduced form utility function. However, there has been very little attention paid to how the choice of utility function affects structural predictions. In particular, there is an interesting feature embedded in a utility function–whether or not additional links cause positive or negative externalities to uninvolved vertices. This paper seeks to show that the choice of underlying utility function and the treatment of externalities is of critical importance to which networks are optimal.

Jackson and Wolinsky's seminal paper *A Strategic Model of Social and Economic Networks* [1996, JET], hereafter **JW**, introduced two reduced form models and solved for the socially optimal network in each. These models can be described by the utility an individual gets from a graph.

JW's Connections Model

$$u_i(G) = \sum_{j \neq i} \delta^{d(i,j)} - c * d_{v_i}$$

where $0 \leq \delta \leq 1$, $d(i,j)$ is the length of the shortest path between v_i and v_j, and d_{v_i} is the number of agents v_i has an edge with.

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[1] See Granovetter (1973, 1995), Rees (1966), and Montgomery (1991).

[2] Jackson (2003) is an excellent survey.

JW's Co-author Model

$$u_i(G) = \sum_{j:e_{i,j} \in E(G)} [\frac{1}{d_{v_i}} + \frac{1}{d_{v_j}} + \frac{1}{d_{v_i} d_{v_j}}]$$

The Connections Model has a very nice solution space. If we measure aggregate utility as the sum of individual utilities, then the socially optimal graph is either empty, a star, or complete. A star is a network with one central agent involved in every connection[3]. Moreover, for appropriate link costs, these graphs will be pairwise stable. The Co-author Model has a much less interesting solution space. For an even number of participants, 2N, the socially optimal graph is simply N pairs. It is possibly for this reason that the Connections Model (or derivations of it) appears much more frequently in the subsequent literature.

The star turns out to be optimal for a wide class of models. Bloch and Jackson (2007) define a utility function to be *distance based* if there exist c and f such that

$$u_i(G) = \sum_{j \neq i} f(d(i,j)) - c * d_{v_i}$$

where $d(i,j)$ is the number of links in the shortest path between vertices i and j, f is a nonincreasing function, and c is a cost per link. They demonstrate that the unique non-trivial efficient network is the star network. The star formation appears frequently in the literature under a variety of different utility functions. Given their result, it is not surprising that the common ingredient among these models is they are distance based.

The fact that the star is optimal for such a wide class of utility functions makes a compelling case for it as a real world model. However, there are at least two points of concern. First, in the star every vertex's payoff is strictly increasing in the size of the network, moreover at an increasing rate. Thus, not only do these models predict the optimal network to be a star but the

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[3] See Figure 1 on page 8 for an example.

largest star possible. However, we expect real-world networks to lose their value and start to break down as they get too large. Moreover, the entire world organized as a star is clearly not ideal for any situation. Another concern is that, although star networks are widely predicted by economic theory, they are not commonly observed in practice. We may observe star-like networks such as airlines' hub-and-spoke systems, but I know of no true star networks.

A distance based model is an environment where all externalities must be positive. If two agents form a link, then they weakly decrease the distance all other agents are apart and weakly increase the number of other agents they are connected to. There are at least two important considerations that a model with only positive externalities does not capture. First, in most networks we expect there to be congestion issues. This is especially true if we are discussing a computer network, but congestion also occurs in most economic networks. For example, **JW's** co-authors model is meant to capture that as the number of co-authors you have increases, you have less time to devote to each one. We would expect the same thing to occur in a network of friends, a communications network, a network of business associates, etc. A star should be especially prone to congestion issues as there is a clear bottleneck, the central node.

A second consideration is that most economic networks involve competition among agents. In a network of buyers and sellers, an additional link literally means increased competition. In a network of gamblers exchanging private information about a horse race, the value of the information decreases with each additional person who learns the inside tip. When an MBA student talks about networking, they are referring to contacts to help them get a job. One can imagine it would be very helpful to have a friend forward your resume to her boss. However, the value of this is substantially decreased if she also forwards the resume of twenty other friends.[4]

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[4] Calvo-Armengol and Jackson (2004) captures this effect. In their model, when

A natural way of modeling a network where an agent's payoff is adversely affected by competition or congestion is through a degree based utility model. I will define a utility function to be *degree-based* if there exists a ϕ such that

$$u_v(G) = \sum_{w \in N(v)} \phi(d_w)$$

where ϕ is non-increasing and d_w is the number of direct relationships w has[5]. In this environment, externalities can only be negative. A new connection in a network weakly increases the degree of each of an agent's neighbors, so if the agent is not directly involved with the new link, her payoff must weakly decrease. This paper will completely characterize the socially optimal and equilibrium set of networks for degree based utility functions.

First, I will show that the regular network is socially optimal for any degree based utility function[6]. Next, I will introduce two new equilibrium concepts which extend the traditional notion of pairwise stability. Under pairwise stability, an agent is able to biilaterally add an edge or unilaterally drop an edge. Strong pairwise stability, a natural extension of pairwise stability, allows an agent to both drop an edge and add another concurrently. Under strong pairwise stability with transfers, an agent is also able to transfer utility to her immediate neighbors. I will be able to completely characterize the set of strongly-pairwise-stable networks for any degree based utility function. Finally, with only a mild assumption on the consistency of externalities, I will be able to show that a network is strongly pairwise stable with transfers if and only if it is the socially optimal regular graph.

These results for degree based utility functions are not only interesting in

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an employed agent hears about a job, she randomly picks an unemployed acquaintance to pass the job to. In this model, edges have negative externalities as an acquaintance adding a connection decreases the chances you will be randomly chosen in the event you are unemployed.

[5] The notation $w \in N(v)$ will be explained more fully in the section on Graph Theory terminology, but it means any vertex w that has a direct edge with v.

[6] A network is regular if all agents have the same number of connections

their own right but especially as a contrast to the positive externalities environment. If we use as our metric maximum degree minus minimum degree, then the star and a regular graph are as different as two graphs can possibly be. It is interesting that two very natural classes of utility functions can lead to such strikingly different optimal and equilibrium networks. As such, this should be taken as a note of caution for researchers using a reduced form utility function to model a social network. The choice of utility function and more broadly the treatment of externalities in these networks are of critical importance to the predictions of the model.

As one of the first papers in networks and specifically one that characterized the solution space of two reduced form models, **JW** has greatly influenced the models in subsequent papers. The solution space for the connections model, the star, is quite interesting whereas the solution space for the co-authors model is trivial. Possibly for this reason, most subsequent papers are based roughly on the connections model. As a result they are positive externalities models. This is unfortunate as most situations of interest to economists involve competition and thus exhibit negative externalities. The final contribution of this paper is to propose and solve a negative externalities model that both has a more interesting solution space than the co-authors model and is a more natural counterpoint to the connections model.

This paper is closely related to Bloch and Jackson (2007). The two papers, in conjunction, are able to completely characterize the two most intuitive, general classes of network models. The focus of Bloch and Jackson is on network formation. They introduce several games in which players make decisions about both link formation and transfers to other agents in the network. Their paper highlights the importance of externalities and demonstrates a reasonable way in which agents might overcome them. The equilibrium concept I introduce, strong pairwise stability with transfers, is a core concept which complements the network formation games introduced in their paper.

This paper is most closely related to Jackson and Wolinsky (1996). They present and solve two reduced form models. Their first model, the connections model, is a distance based model, while their second, the co-authors model, is essentially a degree based model. Therefore, the results presented here in conjunction with the results in Bloch and Jackson (2007) can be viewed as a generalization of **JW**.

There are at least two other papers, Currarini (2002) and Goyal and Joshi (2002), that look at externalities in networks; however, both papers take a substantially different modeling approach than this paper. Currarini (2002) focuses on the partition of vertices into connected components. In Currarini's model the value of a network depends only on this partition. Externalities are defined by whether the value of a network increases or decreases when the partition becomes finer. The network matters in that it determines the partition, but in Currarini's framework the role of network architecture is minimized. Goyal and Joshi's (2002) approach is more similar to this paper but still substantially different. They examine two interesting models where agents have varying degrees in equilibrium. The focus of their paper is how differing degrees affect agent payoff. Externalities, in the form of whether or not agents are strategic substitutes or compliments, end up being crucial to solving their models, but their two games are not an attempt to actually model externalities. In fact, strategic complementarities are defined specifically in terms of the particular payoff function they use and their concept is not readily generalizable. In contrast, this paper's primary aim is to model network externalities in as general a way as possible without diminishing the role of network architecture.

The remainder of this article is organized as follows. Section 2 provides a brief overview of graph theory terminology. Section 3 introduces degree based utility functions and completely characterizes the socially optimal and equilibrium networks for any degree based utility function. Section 4 introduces and solves a specific reduced form utility function. Section 5 concludes, and the Appendix provides complete proofs.

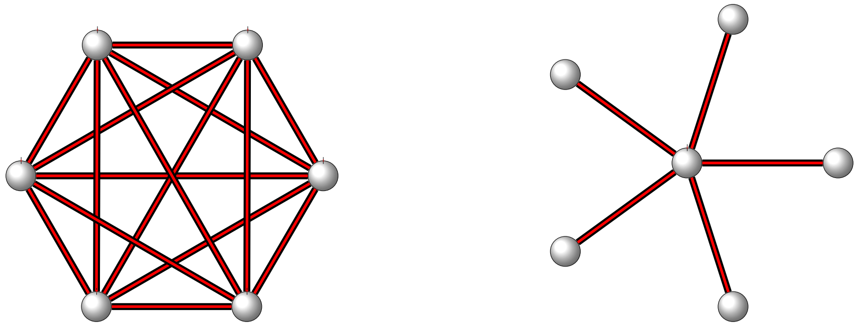
2 Graph Theory Terminology

This paper uses no theoretical results from graph theory, but I will borrow from its terminology to facilitate discussion. I will represent a network as a graph where vertices represent agents and an edge represents a relationship between the two agents. All edges are undirected.

The following are some definitions with corresponding notation:

- E(G) is the set of edges in a graph G. $e_{u,v} \in E(G)$ is an edge between vertices u and v.
- V(G) is the set of vertices in G.
- u and v are *adjacent* if $e_{u,v} \in E(G)$. $u \leftrightarrow v$ indicates u and v are adjacent.
- The *neighborhood* of v is the set of vertices adjacent to v. Symbolically, $N(v) = \{u \in V(G) | e_{u,v} \in E(G)\}$.
- The *degree* of v is the number of vertices v is adjacent to. Symbolically, $d_v = |N(v)|$.
- G is *complete* if all vertices are adjacent.
- A *star* is a graph with one center vertex, in which all remaining vertices are adjacent only to the center.
- A *path* between u and v is a set of edges $\left\{e_{v_1,v_2}, e_{v_2,v_3}, e_{v_3,v_4}, \ldots, e_{v_{n-1},v_n}\right\}$ such that $v_1 = u$ and $v_n = v$.
- u and v are *connected* if there exists a path between them. The *components* of a graph are its maximal connected subgraphs. Note that the connection relation is transitive, symmetric, and reflexive, so it is an equivalence relation. The equivalence classes of the connection relation are the connected components.

Figure 1: A complete graph and a star.



This paper assumes the number of vertices are fixed. Therefore, a graph is completely characterized by its set of edges. As a result, I will slightly abuse notation and use G and E(G) interchangeably. For example, $G \cup e_{u,v}$ represents the new graph created by adding an edge between u and v.

3 Degree Based Utility Functions

In this section I will introduce a class of models which is a natural counterpoint to the more widely studied distance based models. A utility function is *degree-based* if there exists a ϕ such that

$$u_v(G) = \sum_{w \in N(v)} \phi(d_w)$$

. A particularly desirable feature of degree based utility models is that they are interesting even when links are costless to form. For any positive externalities model, if links are costless then the complete network must be socially optimal as adding links can not harm the agents but may help them. As a result, positive externalities models are only interesting to study when links are costly. However, there are situations where incurring a cost for a link does not have a clear interpretation. For example, it is not clear how an agent incurs a cost that is completely separate from the benefit of having a friendship. Time spent maintaining a business relationship or even expenses incurred “wining and dining” a potential partner might reasonably be considered costs, but it is difficult to apply this to a model of personal relationships. After all, having someone to spend time with and someone to go to costly activities such as dinners or baseball games are some of the benefits, not costs, of having a friend. Therefore, it is particularly attractive that we do not need separate costs for links in order to make a degree-based utility function nontrivial.

The star does not perform as well with a rival utility function. The perimeter vertices only get utility from their immediate neighbor, the center of the star. However, the center is connected to so many vertices (the maximum possible) that its value has decreased. Moreover, we do not expect the star to be an equilibrium of a network game as two perimeter vertices would do better by dropping their connection to the center agent and forming a link to each other. In this environment, a more symmetric graph does better socially and is more likely to be pairwise stable. A regular graph, where all agents have the same number of connections, is the natural place to look. Unfortunately, regular graphs do not always exist.[7] However, a regular graph always exists when there is an even number of vertices. I will define a new class of graphs which exist regardless of the parity of the number of agents.

Definition 1. *Let* $\overline{d} = \max\{d(v) : v \in V(G)\}$ *and* $\underline{d} = \min\{d(v) : v \in V(G)\}$. *Then:*

1. *A graph is* ***nearly-regular*** *if* $(\overline{d} - \underline{d}) \leq 1$.
2. *A graph is* ***nearly-n-regular*** *if* $(n-1) \leq \underline{d} \leq \overline{d} \leq n$.

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[7] For example, if we have an odd number of vertices, we can not have a $(2a+1)$-regular graph. Since every edge contributes two to the sum of all vertex degrees, the total sum of degrees must be even. An odd-regular graph with an odd number of vertices would have an odd total degree sum which is not possible.

The next proposition completely characterizes the set of socially optimal networks when there is an even number of vertices. I will be able to simplify this characterization once I impose a mild assumption on the consistency of externalities.

Proposition 1. *Suppose there is an even number of agents. A network G is socially optimal if and only if for every vertex v, $d_v \in \arg\max x\phi(x)$. In particular, for any $n \in \arg\max x\phi(x)$, all n-regular graphs are socially optimal.*

Proof. Each agent receives a payoff *from* her neighbors and contributes utility *to* her neighbors. As an accounting identity, the sum of what every agent receives must equal the sum of what every agent contributes. In particular

$$U(G) = \sum_{i=1}^{N} u_i(G) = \sum_{i=1}^{N} \sum_{v_j \in N(v_i)} \phi(d_{v_j}) = \sum_{i=1}^{N} d_{v_i}\phi(d_{v_i}) \tag{1}$$

Let $n \in \arg\max x\phi(x)$. Since we have an even number of vertices, an n-regular graph exists[8]. Pick any n-regular graph H. By Equation 1, H must be socially optimal. Moreover, if J is a network with a vertex v such that $d_v \notin \arg\max x\phi(x)$, then $U(J) < U(H)$.

□

3.1 Pairwise Stability

Pairwise stability is the standard equilibrium concept in Network Theory. Intuitively, it says no agent wishes to unilaterally drop one of her connections,

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[8] To see this, label the vertices 0 through $|V(G)| - 1$. If n is even, then connect vertex i to vertices $i \pm j mod(|V(G)|)$, for $1 \leq j \leq \frac{n}{2}$. If n is odd, then connect vertex i to vertex $i + \frac{|V(G)|}{2} mod(|V(G)|)$ and to vertices $i \pm j mod(|V(G)|)$, for $1 \leq j \leq \frac{n-1}{2}$

and no two agents wish to bilaterally add a connection. More formally:

Definition 2. *A graph G is pairwise stable if:*

1. *If $e_{i,v} \in E(G)$, then $u_i(G) > u_i(G - e_{i,v})$ and $u_j(G) > u_j(G - e_{i,v})$.*
2. *If $e_{i,v} \notin E(G)$, then either $u_i(G) > u_i(G + e_{i,v})$ or $u_j(G) > u_j(G + e_{i,v})$.*

So far in this section we have assumed that links are costless. However, we must add link costs in order to make pairwise stability interesting. Otherwise, as long as ϕ is strictly positive, the only pairwise stable graph is the complete graph. For the remainder of this subsection, I will assume $u_v(G) = \sum_{w \in N(v)} \phi(d_w) - c \cdot d_v$. Moreover, to avoid any nuisances, I will assume $c \neq \phi(n)$ for any $n \in \mathbb{N}$. This assumption is without loss of generality since one can perturb c by an arbitrarily small ϵ.

This next structure appears several times so I will explicitly define it.

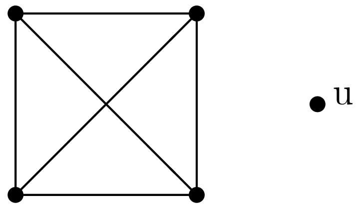
Definition 3. *G is a **maximal nearly-n-regular graph** if it is nearly n regular and there does not exist a nearly-n-regular graph G' such that $E(G) \subset E(G')$.*

Proposition 2. *Let $n = \max\{x \in \mathbb{N} | \phi(x) > c\}$. If utility is rival, then all maximal nearly-n-regular graphs are pairwise stable.*

Proof. Let G be any maximal nearly-n-regular graph. Look at any vertices u and v such that $e_{u,v} \notin E(G)$. The maximality of G implies at least one u or v must have degree n. Without loss of generality, assume $d_v = n$. Then $u_i(G + e_{u,v}) - u_i(G) = \phi(n + 1) - c < 0$ by the maximality of n. Similarly, look at any vertices u and v such that $e_{u,v} \in E(G)$. $u_i(G) - u_i(G - e_{u,v}) \geq \phi(n) - c \geq 0$. Therefore G is pairwise stable. □

Pairwise stability is a fairly weak concept. Pairwise stability only allows a vertex to unilaterally drop a connection or to bilaterally add a connection,

Figure 2: An undesirable pairwise stable graph



but not both. For example, if $\max\{x \in \mathbb{N} | \phi(x) \geq c\} = 3$, then the graph in Figure 2 is pairwise stable since u will be unwilling to add an edge with any vertex that already has degree 3. However, this is unsatisfying as any of the vertices in the 4-clique would be happy to *exchange* one of their edges for an edge with u. Similarly, as long as the vertex is willing to drop one of her edges, u would be happy to form an edge with any vertex in the 4-clique. This leads to a new solution concept.

Definition 4. *A graph G is* ***Strongly Pairwise Stable*** *if*

1. *G is pairwise stable*
2. *There does not exist a* $u, v, w \in V(g)$ *such that* $u_u(G') > u_u(G)$ *and* $u_v(G') > u_v(G)$ *where* $G' = G + e_{u,v} - e_{v,w}$*.*

With this stronger solution concept, we are able to completely characterize the set of strongly pairwise stable graphs.

Proposition 3. *Let* $n = \max\{x \in \mathbb{N} | \phi(x) > c\}$ *and suppose* ϕ *is strictly decreasing. Then G is strongly pairwise stable if and only if G is a maximal nearly-n-regular graph.*

Proof. It is straightforward to verify that a maximal nearly-n-regular graph is strongly pairwise stable. To prove the other direction, look at any strongly pairwise stable graph G. First note that $\max\{d_v | v \in V(G)\} = n$ since otherwise G would not be pairwise stable.[9] Suppose for contradiction that G is not nearly-regular. Then there exists a u and v such that $d_v - d_u \geq 2$. Let $w \in N(v) \setminus N(u) \neq \emptyset$ and let $G' = G + \{e_{u,w}\} - \{e_{v,w}\}$. Note that $u_w(G') > u_w(G)$ since $d_u + 1 < d_v$. Moreover, $u_u(G') > u_u(G)$ since $d_w \leq n$ and $\phi(n) > c$. Therefore G is not strongly pairwise stable, a contradiction. Since G has max degree n and is nearly-regular, it must be nearly-n-regular. If G is not maximal, then there are two non-adjacent vertices of degree n-1 and therefore G is not pairwise stable. □

3.2 Equilibrium with Transfers

In this section I will introduce a new core concept which generalizes pairwise stability to allow agents to make transfers. This complements the games introduced in Block and Jackson (2007). They define several new network games which extend the traditional network games to allow players to make financial transfers. In their paper they make a distinction between who an agent is able to make a transfer to and on what an agent is able to condition this transfer payment. In the direct transfer game, an agent can only make a transfer for a link she is directly involved with. In the indirect transfer games she can only make demands on her own relationships, but she is free to subsidize *any* relationship. In their standard game, a transfer is conditional only on a link forming, but in their game with contingent transfers, an agent can condition a payment on the entire network structure.

Transfers are a natural concept in a social network. We often see situations where maintaining the relationship is more important to one of the agents than the other. In the business world, one person may be in a position of

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[9] If the maximum degree is greater then n, some vertex would want to drop an edge. If the max degree is less then n, any unconnected vertices would be better off adding an edge.

greater power or may simply have more connections than the other. In the academic world, the marginal value of publishing an article should decrease with the total number of publications an author has. Therefore, even if two co-authors are equally talented, the relationship should be relatively more valuable to the person with fewer publications. In such a situation, it is natural for a transfer to occur. The person to whom the relationship is more valuable will have the incentive to put forth more effort into the project or else might agree to do the less desirable aspects of the work.

I propose a new core concept which generalizes pairwise stability to allow agents to make transfers to the people they have a relationship with. I only allow direct transfers as I find indirect transfers much less natural. A transfer between two agents with a relationship can be non-pecuniary; however, there is less flexibility when the transfer is between two agents who are not linked. It is hard to interpret an indirect transfer as anything other that a monetary payment. In some situations this may be perfectly natural, but especially with social networks, direct monetary transfers occur infrequently. An academic can do many things to either ease the burden on her coauthor or make the relationship more valuable, but it would be strange for her to pay an academic to *not* collaborate with any of her coauthors. Despite this restriction, we will still be able to reach some powerful conclusions.

More formally, the game consists of a graph G and a matrix of transfers T. In the transfer matrix, t_{ij} represents the transfer from agent i to agent j. An agent can only make a transfer to someone she has a direct relationship with. Therefore, $t_{ij} > 0$ only if $e_{ij} \in G$. To avoid ambiguity, I will require that $t_{ij} = -t_{ji}$ as we care only about the net transfer.

Therefore, given a network G and transfers T, agent i receives a payoff of

$$\pi_i(G,T) = \sum_{v_j \in N(v_i)} [\phi(d_{v_j}) + t_{ji}] \tag{2}$$

Individually, an agent should be able to drop any of her edges and change the transfers she offers. Two agents should also be able to form a link if they so desire. Anytime an agent changes her edges or transfers, she alters the payoff of her neighbors and, therefore, potentially jeopardizes these relationships. However, if two agents are able to move bilaterally to establish a mutually beneficial relationship and to adjust their transfers so that *all* of their neighbors are better off, then they do not jeopardize these relationships and surely the network is in disequilibrium.

Definition 5. *Given a network G with transfers T, agent v_i* ***blocks*** $< G, T >$ *if there exists an agent v_j*[10]*, subsets $A \subseteq N(v_i)$, $B \subseteq N(v_j)$, and transfers T' on the set $\{v_i, v_j, N(v_i) \setminus A, N(v_j) \setminus B\}$ such that:*

$$\pi_x(G', T + T') > \pi_x(G, T)$$

for every $x \in \{v_i, v_j, N(v_i) \setminus A, N(v_j) \setminus B\}$ where $G' = G \cup e_{i,j} \setminus \{e_{i,k} : k \in A\} \setminus \{e_{j,k} : k \in B\}$.

This definition is purely a generalization of strong pairwise stability. An agent can drop any edge, add an edge (as long as the other agent wishes to as well), or do both simultaneously. If the agent is also able to compensate all the agents she remains in a relationship with so that they are made better off by the change, then surely we are in disequilibrium.

Definition 6. *A network G is* ***strongly pairwise stable with transfers*** *if there exists transfers T such that no agent blocks $< G, T >$. In such a case, we say the transfers T* ***support*** *G.*

When it is clear from context that it is a network with transfers, I will just say strongly pairwise stable instead of strongly pairwise stable with transfers. With a mild regularity assumption on externalities, I will prove that the only

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[10] We allow $v_j = \emptyset$ in which case we interpret $G \cup e_{i,j} = G$.

network that is strongly pairwise stable with transfers is the socially optimal, nearly-regular network.

Whenever an edge is added to an network, there is a social trade off. There is a direct benefit to the two agents forming the relationship but at a cost of a decreased payoff to all the agents they already share an edge with. This trade-off is captured by the function:

$$to(x) = \phi(x+1) - x(\phi(x) - \phi(x+1)), 1 \leq x \leq |V(G)| - 2$$

When an agent with degree x forms a new edge, she is contributing $\phi(x+1)$ to the new neighbor but at a cost of $\phi(x) - \phi(x+1)$ to the x many agents she was previously connected to. This motivates a regularity condition on externalities.

Definition 7. *The externalities in a network are **consistent** if* $to(x)$ *is decreasing.*

Definition 8. *Externalities are **weakly consistent** if any of the following conditions hold:*

1. $to(x) > 0$.
2. $to(x) < 0$.
3. *There exists a integer* M *such that* $to(x) \geq 0$ *for every* $x \leq M$ *and* $to(x) < 0$ *for every* $x > M$.

*For each respective case, we define the **threshold** of* $to(x)$ *to be:*

1. $|V(G)| - 1$
2. *1*
3. M

$to(x)$ is decreasing if $\phi(x) - \phi(x+1)$ is increasing. Therefore, if ϕ is concave, then externalities are consistent. However, this condition is much weaker than concavity. Of course, if externalities are consistent then they are weakly consistent.

I will assume that if $to(x)$ is weakly consistent with threshold M with $1 < M < |V(G)| - 1$, then $to(M) = 0$. This does not change the results in any significant way, but it does make the proofs cleaner. Moreover, if $to(M) \neq 0$, then we can replace $\phi(x)$ by $\phi'(x)$ where $\phi'(x) = \phi(x) - to(M)$. Now:

$$\begin{aligned} to_{\phi'} &= \phi'(x+1) - x(\phi'(x) - \phi'(x+1)) \\ &= \phi(x+1) - to(M) - x(\phi(x) - to(M) - \phi(x+1) + to(M)) \\ &= to_{\phi}(x) - to(M) \end{aligned}$$

and therefore $to_{\phi'}(M) = 0$.

With this mild assumption on externalities, we can completely classify the set of socially optimal networks. In a surprising result, once agents are able to make transfers to their neighbors, these networks will also be the only strongly pairwise stable networks.

Proposition 4. *Suppose externalities are weakly consistent.*

1. *When $1 < M < |V(G)| - 1$, then G is socially optimal if and only if G is nearly-(M+1)-regular.*
2. *When $M = |V(G)| - 1$, then the complete network is the unique socially optimal network.*
3. *When $M = 1$, then the unique socially optimal network is the trivial network*[11]*.*

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[11] The trivial network consists of $\frac{|V(G)|}{2}$ many pairs if $|V(G)|$ is even and $\frac{|V(G)-3|}{2}$

I will prove Proposition 4.1 here. The remaining cases, which are more technical, are proved in the Appendix.

Proof. We know from the previous section that G is optimal if and only if every vertex has degree from the $argmax x\phi(x)$. Note that

$$\begin{aligned} to(x) &= \phi(x+1) - x(\phi(x) - \phi(x+1)) \\ &= (x+1)\phi(x+1) - x\phi(x) \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{2\leq i\leq x} to(i-1) &= \sum_{2\leq i\leq x} [i\phi(i) - (i-1)\phi(i-1)] \\ &= x\phi(x) - \phi(1) \end{aligned}$$

since $\sum_{2\leq i\leq x}[i\phi(i) - (i-1)\phi(i-1)]$ is a telescoping series. In particular,

$$x\phi(x) = [\sum_{2\leq i\leq x} to(i-1)] + \phi(1) \tag{3}$$

Since, $to(x) \geq 0$ for every $x \leq M$ and $to(x) < 0$ for every $x > M$, $x\phi(x)$ is maximized at $x = M+1$. Note that

$$\begin{aligned} (M+1)\phi(M+1) &= [\sum_{2\leq i\leq (M+1)} to(i-1)] + \phi(1) \\ &= to(M) + [\sum_{2\leq i\leq (M)} to(i-1)] + \phi(1) \\ &= 0 + [\sum_{2\leq i\leq (M)} to(i-1)] + \phi(1) \\ &= M\phi(M) \end{aligned}$$

---

many pairs plus the three remaining vertices connected as a path if $|V(G)|$ is odd. See Figure 3 on page 26 for an example.

Where the second to last equality follows from our assumption that $to(M) = 0$. Since $to(x) \neq 0$ for every $x \neq M$, $argmax(to(x)) = \{M, M+1\}$. As mentioned previously, a nearly regular graph always exists. Therefore, a network is optimal if and only if it is nearly-(M+1)-regular.

$\square$

I will now prove the main theorem of this section.

Proposition 5. *Suppose externalities are weakly consistent.*

1. *When $1 < M < |V(G)| - 1$, G is strongly pairwise stable with transfers if an only if G is nearly-(M+1)-regular.*
2. *When $M = |V(G)| - 1$, the complete network is the unique network that is strongly pairwise stable with transfers.*
3. *When $M = 1$, the unique network that is strongly pairwise stable with transfers is the trivial network.*

This proposition is proved with a sequence of lemmas. Complete proofs are given in the Appendix, but I will list the lemmas and give the intuition to the proof here.

Lemma 1. *Suppose $1 < M < |V(g)| - 1$ and let G be strongly pairwise stable. If there exists a vertex u with degree less than M, then u is adjacent to every vertex with degree less than or equal to M.*

Intuition. If there are two non-adjacent agents with degree less than the threshold, then adding an edge is socially beneficial as the trade-off for both agents is positive. Moreover, all the social gains are realized by the two agents. Since it is socially beneficial, they benefit enough to be able to compensate all of their neighbors and still improve their payoff.

Lemma 2. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be strongly pairwise stable. If there exists a vertex* u *with degree greater than* $M + 1$*, then* u *is not adjacent to any vertex with degree greater than or equal to* $M + 1$*.*

Intuition. It is socially beneficial for the two agents with degree greater than the threshold to cut their relationship. Their neighbors, who receive all the benefit from the two agents cutting an edge, are willing and able to increase their transfers by enough to make it in u or v's best interest to cut the edge. We have to be a little more careful than in Lemma 1 since there may be transfers between the two agents which would affect their willingness to drop the edge.

Lemma 3. *Suppose* T *supports a network* G*. Then for every two agents* i *and* j *such that* $e_{i,v} \in G$*,*

$$t_{ij} \leq \phi(d_j) - (d_i - 1)(\phi(d_i - 1) - \phi(d_i))$$

Proof. v_i has $d_i - 1$ many neighbors who would be willing to pay up to $\phi(d_i - 1) - \phi(d_i)$ for i to sever her relationship with j. i receives a benefit of $\phi(d_j) - t_i, j$ from her relationship with j, so if $\phi(d_j) - t_{ij} < (d_i - 1)(\phi(d_i - 1) - \phi(d_i))$ then i and all her remaining neighbors do strictly better if i drops her relationship with j and accepts a transfer of $\phi(d_i - 1) - \phi(d_i) - \epsilon$ from each of her remaining neighbors. ☐

Lemma 4. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be strongly pairwise stable. If there exists a vertex* u *with* $d_u > M + 1$*, then all of* u*'s neighbors are adjacent.*

Intuition. If there is an agent u such that $d_u > M + 1$, $v, w \in N(u)$, but $e_{vw} \notin G$, then v and w are better off dropping their edges with u and creating an edge between themselves. They do not need to compensate their neighbors for such a move as their degree has not changed. We know from Lemma 2 that d_v and d_w are both less than $M + 1$, so, both v and w are made better off as they each have degree less than u. The only thing we have to be careful about is that u may be subsidizing her relationship with either or both of the agents, and when they sever the relationship with u they forgo the subsidy. Lemma 3 sets a bound on the transfer from u to v or w that ensures it will always be in v and w's best interest to forgo the transfer and establish a relationship with each other.

Lemma 5. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be strongly pairwise stable. No vertex in* G *has degree greater than* $M + 1$*.*

Intuition. This is a pigeonhole argument. Suppose for contradiction there is a vertex u with $d_u > M + 1$. By Lemma 2, every neighbor of u must have degree less than or equal to M. By Lemma 4, all neighbors of u must be adjacent. However, there are at least $M + 1$ neighbors of u. All are adjacent to the other neighbors of u (there are at least M other neighbors of u) plus u itself. Therefore, all neighbors of u must have degree at least $M + 1$, a contradiction.

Lemma 6. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be socially optimal. No vertex in* G *has degree less than* M*.*

Intuition. If there exists a vertex u with $d_u < M$, we know from Lemma 1 that u must be adjacent to all vertices with degree less than or equal to M. We can establish that there must be two vertices, v and w that are adjacent to each other, but neither of which are adjacent to u. Either of these agents would be better off dropping the edge with the other and instead establishing an edge with u. Moreover, this will be socially beneficial as both v and w must have degree $M + 1$ and socially we are indifferent whether or not they are adjacent. However, since $d_u < M$, we strictly prefer that u create a new relationship. Since this switch is socially beneficial and all the gains are realized by u and the vertex that switches, they will be able to compensate u's neighbors for their decreased payoff. Again, we have to be careful about transfers between v and w, but only one can be receiving a positive transfer from the other.

Lemma 7. *Suppose* $1 < M < |V(g)| - 1$. *If G is nearly-(M+1)-regular, then G is strongly pairwise stable with transfers.*

Proof. Let G be any nearly-(M+1)-regular graph. Define a set of transfers T by:

$$t_{uv} = \begin{cases} 0 & d_u = d_v \\ \phi(M) - \phi(M+1) & d_u = (M+1), d_v = M \\ \phi(M+1) - \phi(M) & d_u = M, d_v = M+1 \end{cases}$$

Every agent with degree M receives a total payoff of $M\phi(M)$ and every agent with degree $M+1$ receives a payoff of $(M+1)\phi(M+1)$. Since $to(M) = 0$, $M\phi(M) = (M+1)\phi(M+1)$.

A nearly-(M+1)-regular graph is optimal, so adding an edge cannot increase social payoff. Since all the benefits are captured by the two agents adding an edge and all costs are incurred by their neighbors, it is not possible for the two agents adding the edge to make all their neighbors better off. Similarly, an agent u has no wish to delete one of her edges. u's remaining neighbors receive all the benefit, while u incurs all the costs. Since the costs are greater than or equal to the benefits (the original graph was socially optimal), u's remaining neighbors will not be able to compensate u so that all are better off. Finally, two vertices u and v can not do better by each dropping an edge and creating an edge with each other. The new relationship is worth at most $\phi(M)$ which is exactly what they received from their previous relationship. □

Lemma 5 and Lemma 6 establishes the nearly-(M+1)-regularity is necessary for strong pairwise stability with transfers. Lemma 7 establishes that being nearly-(M+1)-regular is sufficient as well. This is a surprising and powerful result. In a network of relationships, an agent should be able to sever any ties it chooses and establish new ties when it is mutually desirable. Moreover, there should always be informal ways an agent can exert effort that is costly for herself but makes the relationship more beneficial for a partner. My result establishes that if this is case, then the only network which will be an equilibrium is the socially optimal network.

4 A Reduced Form Utility Model

A particular degree based utility function of interest is:

$$u_i(G) = w_i + \sum_{i \leftrightarrow j} \gamma^{d_{v_j}} w_{i,j} - \sum_{i \leftrightarrow j} c_{i,j}, \text{ where } 0 \leq \gamma \leq 1. \tag{4}$$

Recall JW's Connections Model is:

$$u_i(G) = w_i + \sum_{j \neq i} \delta^{t_{i,j}} w_{i,j} - \sum_{i \leftrightarrow j} c_{i,j}$$

where $t_{i,j}$ is the length of the shortest path between v_i and v_j.

As mentioned before, the Connections Model has only positive externalities. The co-authors model is the negative externalities model JW examine, but the utility function presented in Equation 4 is a more natural negative externalities counterpoint to the connections model. A vertex only gets utility from its neighbors, and this utility is a decreasing function of each neighbor's degree. This also fits **JW**'s motivation for the co-authors model. The benefit to working with a colleague is decreasing in the number of co-authors she has as she will have less time to devote to your project.

With our results from Section 3, we can quickly solve for the symmetric version of Equation 4. Let

$$u_i(G) = \sum_{i \leftrightarrow j} \gamma^{d_{v_j}} \tag{5}$$

where $0 < \gamma < 1$. Further, suppose $\gamma = \frac{\tau}{\tau+1}$ for some integer τ.

By assumption:

$$\begin{aligned} to(x) &= (x+1)\gamma^{x+1} - x\gamma^{x} \\ &= \gamma^{x}((x+1)\gamma - x) \\ &= \gamma^{x}(x(\gamma-1)+1) \end{aligned}$$

which is a decreasing function of x. Moreover

$$\begin{aligned} to(\tau) &= \gamma^{\tau}((\tau+1)\gamma - \tau) \\ &= \gamma^{\tau}((\tau+1)\frac{\tau}{\tau+1} - \tau) \\ &= \gamma^{\tau}((\tau - \tau) \\ &= 0 \end{aligned}$$

Therefore all the assumptions of Proposition 3 on page 19 are met.

Proposition 6. *Suppose* $u_i(G) = \sum_{i\leftrightarrow j} \gamma^{d_{v_j}}$. *Then*

1. *G is socially optimal if and only if G is nearly-*$(\tau+1)$*-regular.*
2. *G is strongly pairwise stable with transfers if and only if G is nearly-*$(\tau+1)$*-regular.*

5 Conclusion

Distance based and degree based models are the two most intuitive models of an agent's payoff from a network. While much attention has been paid to distance based models, very little has been paid to degree based models. This paper completely characterizes the set of optimal and stable networks for this natural class of utility functions. The predicted networks are interesting in their own right but especially so when taken in contrast to the optimal networks for distance based models. It is striking that two intuitive models can lead to such dramatically different predictions. In particular, this paper, taken in conjunction with the results from Bloch and Jackson (2007), provides a generalization and simplification of results in the classic networks paper by Jackson and Wolinsky (1996).

6 Appendix - Proofs

The trivial network consists of $\frac{|V(G)|}{2}$ many pairs if $|V(G)|$ is even and $\frac{|V(G)-3|}{2}$ many pairs plus the three remaining vertices connected as a path if $|V(G)|$ is odd.

Proposition 4. *Suppose externalities are weakly consistent.*

1. *When $1 < M < |V(G)| - 1$, then G is socially optimal if an only if G is nearly-(M+1)-regular.*
2. *When $M = |V(G)| - 1$, then the complete network is the unique socially optimal network.*
3. *When $M = 1$, then the unique socially optimal network is the trivial network.*

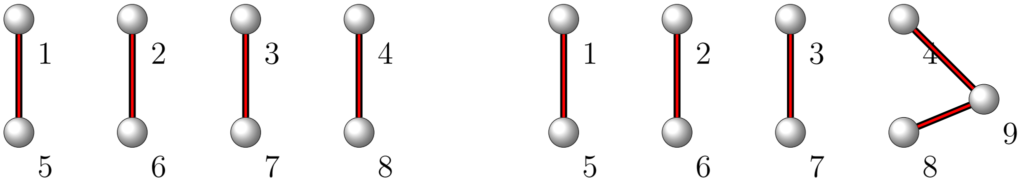
Proof. Proposition 4.2 Let G be any graph that contains two non-adjacent vertices u and v, and suppose the threshold is $|V(G)| - 1$. By the definition of the threshold, $to(x) > 0$. Therefore,

$$\begin{aligned} U(G \cup e_{u,v}) - U(G) &= (d_u+1)\phi(d_u+1) - d_u\phi(d_u) + (d_v+1)\phi(d_v+1) - d_v\phi(d_v) \\ &= to(d_u) + to(d_v) \end{aligned}$$

$$> \quad 0$$

and G can not be optimal. Therefore, the only optimal network is one where all vertices are adjacent.

Figure 3: Trivial graph for an even and odd number of vertices.



Proposition 4.3 We know from Equation 1 on page 10 that

$$U(G) = \sum_{i=1}^{N} d_{v_i}\phi(d_{v_i})$$

and from Equation 3 on page 18 that

$$x\phi(x) = [\sum_{2\leq i\leq x} to(i-1)] + \phi(1)$$

Since $to(x) < 0$ for every $x > 0$, $x\phi(x)$ is strictly decreasing for $x \geq 1$. Therefore, if a 1-regular graph exists, it must be optimal. A 1-regular graph exists when there is an even number of vertices (the trivial graph), but does not when the number of vertices is odd. Since $2\phi(2) > 0$, the trivial graph must be optimal when there is an odd number of agents.

□

Lemma 1. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be strongly pairwise stable. If there exists a vertex* u *with degree less than M, then* u *is adjacent to every vertex with degree less than or equal to M.*

Proof. Suppose for contradiction that G is supported by transfers T and has two non-adjacent vertices u and v with $d_u < M$ and $d_v \leq M$. Let

$$t'_{vx} = \begin{cases} \phi(d_v) - \phi(d_v + 1) + \epsilon & x \in N(v) \\ \phi(d_u + 1) - \phi(d_v + 1) & x = u \end{cases}$$
$$t'_{ux} = \begin{cases} \phi(d_u) - \phi(d_u + 1) + \epsilon & x \in N(u) \\ \phi(d_v + 1) - \phi(d_u + 1) & x = v \end{cases}$$

Then

$$\begin{aligned} \Delta\pi_u(G \cup e_{i,j}, T + T') &= \phi(d_v + 1) + t'_{vu} - \sum_{w \in N_G(u)} t'_{uw} \\ &= \phi(d_v + 1) + [\phi(d_u + 1) - \phi(d_v + 1)] + d_u(\phi(d_u + 1) - \phi(d_u) - \epsilon) \\ &= \phi(d_u + 1) - d_u(\phi(d_u) - \phi(d_u + 1)) - d_u * \epsilon) \\ &= to(d_u) - d_u * \epsilon \\ &> 0 \end{aligned}$$

for ϵ sufficiently small.

For $x \in N_G(u)$

$$\begin{aligned} \Delta\pi_x(G \cup e_{i,j}, T + T') &= \phi(d_u + 1) - \phi(d_u) + t'_{ux} \\ &= \phi(d_u + 1) - \phi(d_u) + \phi(d_u) - \phi(d_u + 1) + \epsilon \\ &= \epsilon \\ &> 0 \end{aligned}$$

Similarly, $\pi_v(G \cup e_{i,j}, T + T') - \pi_v(G, T) > 0$ and $\pi_x(G \cup e_{i,j}, T + T') - \pi_u(G, T) > 0$ for every $x \in N_G(v)$. Therefore, u and v block $< G, T >$, contradicting the assumption that T supports G. □

Lemma 2. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be strongly pairwise stable. If there exists a vertex* u *with degree greater than* $M + 1$*, then* u *is* *not adjacent to any vertex with degree greater than or equal to $M + 1$.*

Proof. Suppose for contradiction that G is supported by transfers T and has two adjacent vertices u and v with $d_u > M + 1$ and $d_v \geq M + 1$. Let $r_{xu} = \phi(d_u - 1) - \phi(d_u) - \epsilon$ for every $x \in N(v) \setminus u$. In other words, each neighbor of u *gives* u $\phi(d_u - 1) - \phi(d_u) - \epsilon$. Similarly, let $s_{vx} = \phi(d_v) - \phi(d_v - 1) + \epsilon$ for every $x \in N(u) \setminus v$.

Then if u drops its edge with v in order to receive transfers R, it loses the benefit from v, $\phi(d_v)$, no longer makes the transfer $t_{u,v}$, and gains the transfers R from each of its remaining neighbors. Specifically,

$$\Delta\pi_u(G \setminus e_{uv}, T + R) = -\phi(d_v) + t_{uv} + (d_u - 1)(\phi(d_u - 1) - \phi(d_u) - \epsilon)$$

Similarly,

$$\Delta\pi_v(G \setminus e_{uv}, T + S) = -\phi(d_u) + t_{vu} + (d_v - 1)(\phi(d_v - 1) - \phi(d_v) - \epsilon)$$

Adding these two equations yields

$$\begin{aligned} \Delta\pi_u(G \setminus e_{uv}, T + R) + \Delta\pi_v(G \setminus e_{uv}, T + S) &= \\ -to(d_v - 1) - to(d_u - 1) + t_{uv} + t_{vu} - \epsilon(d_u + d_v - 2) &= \\ -to(d_v - 1) - to(d_u - 1) - \epsilon(d_u + d_v - 2) &> 0 \end{aligned}$$

for sufficiently small ϵ. The first equality comes from rearranging terms. The second equality follows since $t_{uv} = -t_{vu}$ by definition. The final inequality follows since $d_u - 1 > M$ and $d_v - 1 \geq M$, so by the definition of the threshold, $\phi(d_u - 1) < 0$ and $\phi(d_v - 1) \leq 0$. But, since $\Delta\pi_u(G \setminus e_{uv}, T + R) + \Delta\pi_v(G \setminus e_{uv}, T + S) > 0$, either $\Delta\pi_u(G \setminus e_{uv}, T + R) > 0$ or $\Delta\pi_v(G \setminus e_{uv}, T + S) > 0$.

By construction, $\pi_x(G \setminus e_{uv}, T + R) - \pi_x(G, T) > 0$ for every $x \in N(u) \setminus v$ and $\pi_x(G \setminus e_{uv}, T + S) - \pi_x(G, T) > 0$ for every $x \in N(v) \setminus u$. Since

$$[\pi_u(G \setminus e_{uv}, T + R) - \pi_u(G, T)] + [\pi_v(G \setminus e_{uv}, T + S) - \pi_v(G, T)] > 0$$

at least one of $[\pi_u(G \setminus e_{uv}, T + R) - \pi_u(G, T) > 0$ or $\pi_v(G \setminus e_{uv}, T + S) - \pi_v(G, T) > 0$. Whichever one is greater than zero blocks G, a contradiction. $\square$

In equilibrium, there is a limit to how much an agent is willing to transfer another agent.

Lemma 3. *Suppose T supports a network G. Then for every two agents i and j such that $e_{i,v} \in G$,*

$$t_{ij} \leq \phi(d_j) - (d_i - 1)(\phi(d_i - 1) - \phi(d_i))$$

.

Proof. v_i has $d_i - 1$ many neighbors who would be willing to pay up to $\phi(d_i - 1) - \phi(d_i)$ for i to sever her relationship with j. i receives a benefit of $\phi(d_j) - t_i, j$ from her relationship with j, so if $\phi(d_j) - t_{ij} < (d_i - 1)(\phi(d_i - 1) - \phi(d_i))$ then i and all her remaining neighbors do strictly better if i drops her relationship with j and accepts a transfer of $\phi(d_i - 1) - \phi(d_i) - \epsilon$ from each of her remaining neighbors. $\square$

Lemma 4. *Suppose $1 < M < |V(g)| - 1$ and let G be strongly pairwise stable. If there exists a vertex u with $d_u > M + 1$, then all of u's neighbors are adjacent.*

Proof. Suppose not, and let u be such that $d_u > M + 1$, $v, w \in N(u)$, but $e_{vw} \notin G$. We know from Lemma 2 that $d_v, d_w \leq M$. Since v and w are not adjacent, we know from Lemma 1 that neither v nor w has degree less than M. Therefore, $d_v = d_w = M$.

From Lemma 3

$$t_{uv} \leq \phi(d_v) - (d_u - 1)(\phi(d_u - 1) - \phi(d_u))$$

$$
\begin{aligned}
&= \phi(d_v) - \phi(d_u) + \phi(d_u) - (d_u - 1)(\phi(d_u - 1) - \phi(d_u)) \\
&= \phi(M) - \phi(d_u) + to(d_u - 1)
\end{aligned}
$$

Similarly, $t_{uw} \leq \phi(M) - \phi(d_u) + to(d_u - 1)$. Therefore

$$
\begin{aligned}
\Delta\pi_v(G + e_{vw} \setminus \{e_{uv}, e_{uw}\}, T) &= \phi(d_w) - \phi(d_u) - t_{uv} \\
&= \phi(M) - \phi(d_u) - t_{uv} \\
&\geq \phi(M) - \phi(d_u) - (\phi(M) - \phi(d_u) + to(d_u - 1)) \\
&= -to(d_u - 1) \\
&> 0
\end{aligned}
$$

where the last inequality follows from $d_u > M + 1$ and therefore, $to(d_u - 1) < 0$.

Similarly, $\Delta\pi_w(G + e_{vw} \setminus \{e_{uv}, e_{uw}\}, T) > 0$. Note that since the degree of v and w has not changed, all vertices in $N(v) \cup N(w) \setminus u$ are indifferent between $< G + e_{vw} \setminus \{e_{uv}, e_{uw}\}, T >$ and $< G, T >$. Therefore agents v and w block $< G, T >$ contradicting the stability of G.

□

Lemma 5. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be strongly pairwise stable. No vertex in* G *has degree greater than* $M + 1$.

Proof. This is a pigeonhole argument. Suppose for contradiction there is a vertex u with $d_u > M + 1$. By Lemma 2, every neighbor of u must have degree less than or equal to M. By Lemma 4, all neighbors of u must be adjacent. However, there are at least $M + 1$ neighbors of u. All are adjacent to the other neighbors of u (there are at least M other neighbors of u) plus u itself. Therefore, all neighbors of u must have degree at least $M + 1$, a contradiction. □

Lemma 6. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be socially optimal. No vertex in* G *has degree less than* M*.*

Proof. Suppose for contradiction there exists a vertex u with $d_u < M$. Since $M < |V(G)| - 1$, there exists a v not adjacent to u. By Lemma 1, $d_v > M$. Therefore, by Lemma 5, $d_v = M + 1$. Since v has $M + 1$ neighbors and u has less than M neighbors, there must exist a w which is adjacent to v but not adjacent to u. Repeating the logic above, $d_w = M + 1$. We will demonstrate that u, w, and all of their neighbors can be made better off if u adds an edge with w and w drops it's edge with v.

From Lemma 3

$$\begin{aligned} t_{wv} &\leq \phi(d_v) - (d_w - 1)(\phi(d_w - 1) - \phi(d_w)) \\ &= \phi(M + 1) - (M)(\phi(M) - \phi(M + 1)) \\ &= to(M) \\ &= 0 \end{aligned}$$

Similarly $t_{vw} \leq 0$, therefore $t_{wv} = t_{vw} = 0$. Let $G' = G \cup e_{uw} \setminus e_{vw}$. Then

$$\begin{aligned} \Delta\pi_w(G', T) &= \phi(d_u + 1) - \phi(d_v) - t_v w \\ &= \phi(d_u + 1) - \phi(d_v) \\ \Delta\pi_u(G', T) &= \phi(d_w) \\ \Delta\pi_x(G', T) &= \phi(d_u + 1) - \phi(d_u) \text{ for every } x \in N(u) \\ \Delta\pi_x(G', T) &= 0 \text{ for every } x \in N(w) \end{aligned}$$

Let

$$\begin{aligned} t'_{ux} &= \begin{cases} \phi(d_u) - \phi(d_u + 1) + \epsilon & x \in N(u) \\ \phi(d_v) - \phi(d_u + 1) + \omega & x = w \end{cases} \\ t'_{wx} &= \delta \text{ for every } x \in N(w) \setminus v. \end{aligned}$$

Now

$$\begin{aligned}
\Delta\pi_w(G', T+T') &= \omega - d_w * \delta \\
\Delta\pi_u(G', T+T') &= \phi(d_u+1) - d_u(\phi(d_u) - \phi(d_u+1)) - d_u * \epsilon - \omega \\
&= to(d_u) - d_u * \epsilon - \omega \\
\Delta\pi_x(G', T+T') &= \epsilon \text{ for every } x \in N(u) \\
\Delta\pi_x(G', T+T') &= \delta \text{ for every } x \in N(w)
\end{aligned}$$

Since $d_u < M$, $to(d_u) > 0$, and therefore, u, w and all of their neighbors can be made better off in $G \cup e_{uw} \setminus e_{vw}$. This contradicts the strong pairwise stability of G.

$\square$

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